Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 12

Marius A. Marinescu

Departamento de Teoría de la Señal y Comunicaciones
Área de Estadística e Investigación Operativa
Universidad Rey Juan Carlos

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Outline

Vector-Valued Functions

- Chapter 3 concerns vector-valued functions of two special types:
  1. Continuous mappings of one variable called paths in $\mathbb{R}^n$.

     Functions $\mathbf{x} : I \subseteq \mathbb{R} \to \mathbb{R}^n$,
     where I is an interval

  2. Mappings from (subsets of) $\mathbb{R}^n$ to itself, called vector fields.

     Functions $\mathbf{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$,
     where X is a subset of $\mathbb{R}^n$

Remark

An understanding of both concepts is required later,
when we discuss **line** and **surface integrals**

Definition 1.1

- Let I denote any interval in $\mathbb{R}$.
- Thus, I can be of the form:
  - Bound intervals: $[a, b]$, (a, b), $[a, b)$, or $(a, b]$
  - Unbound intervals: $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$, or $(-\infty, \infty) = \mathbb{R}$
- A path in $\mathbb{R}^n$ is a continuous function:

$$\mathbf{x} : I \subseteq \mathbb{R} \to \mathbb{R}^n$$

- If $I = [a, b]$ for some numbers $a < b$, then the points $\mathbf{x}(a)$ and $\mathbf{x}(b)$ are called the endpoints of the path $\mathbf{x}$.

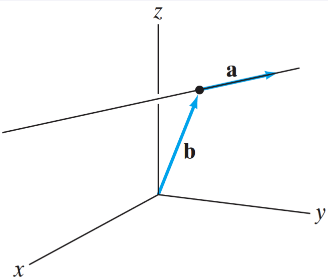
Similar definitions apply if
$I = [a, b), [a, \infty)$, etc.

Example 1

- Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$ with $\mathbf{a} \neq \mathbf{0}$
- Let $\mathbf{x} : (-\infty, \infty) \to \mathbb{R}^3$ be the function given by

$$\mathbf{x}(t) = \mathbf{b} + t\mathbf{a}$$

- Then, this function $\mathbf{x}$ defines the path along the straight line
  - Parallel to $\mathbf{a}$, and
  - Passing through the endpoint of the position vector of $\mathbf{b}$

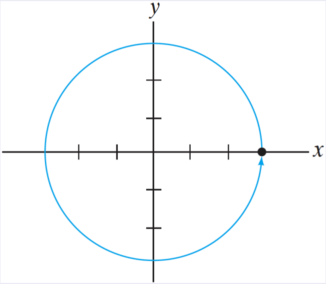


Example 2

- Consider the path $\mathbf{y} : [0, 2\pi) \to \mathbb{R}^2$ given by

$$\mathbf{y}(t) = (3\cos t, 3\sin t)$$

- It can be thought of as the path of a particle that travels once, counterclockwise, around a circle of radius 3



Example 3

- Consider the map $\mathbf{z} : \mathbb{R} \to \mathbb{R}^3$ given by

$$\mathbf{z}(t) = (a\cos t, a\sin t, bt), \quad a, b \text{ constants } (a > 0)$$

- It is called a circular helix

Its projection in the *xy*-plane
is a circle of radius *a*

- The helix itself lies in the right circular cylinder

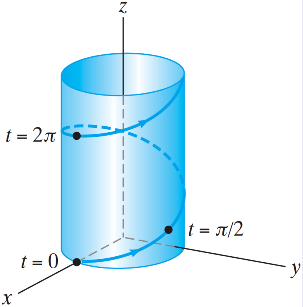
$$x^2 + y^2 = a^2$$

Example 3

$$\mathbf{z}(t) = (a\cos t, a\sin t, bt), \quad a, b \text{ constants } (a > 0)$$

- The helix itself lies in the right circular cylinder

$$x^2 + y^2 = a^2$$

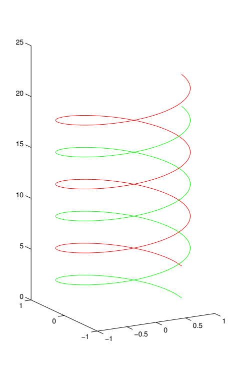


- The value of b determines how tightly the helix twists.

Example 3

$$\mathbf{z}(t) = (a\cos t, a\sin t, bt), \quad a, b \text{ constants } (a > 0)$$

- Using, for instance, MATLAB



Path vs Range

- Note that we distinguish between a path $\mathbf{x}$ and its range or image set $\mathbf{x}(I)$
- A path is a function, a dynamic object

We imagine the independent variable t
to represent time

- The range or image is a curve in $\mathbb{R}^n$

A curve is
a static figure in space

Velocity Vector of the Path

- Thinking a path as a dynamic object, it is natural for us to consider the derivative to be the velocity vector of the path.
- The velocity vector can be denoted as $D\mathbf{x}(t), \mathbf{x}'(t)$ or $\mathbf{v}(t)$.
- Since $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is a function of just one variable, then

$$\mathbf{v}(t) = \mathbf{x}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{x}(t + \Delta t) - \mathbf{x}(t)}{\Delta t}$$

- Thus, $\mathbf{v}(t)$ is the instantaneous rate of change of position $\mathbf{x}(t)$ with respect to t (time).

Velocity Vector of the Path

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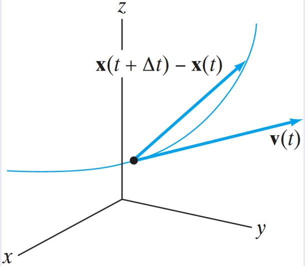


Velocity $\mathbf{v}(t)$ is a vector
tangent to the path at $\mathbf{x}(t)$

Definition 1.2: Velocity, Speed and Acceleration

- Let $\mathbf{x} : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a differentiable path.
- Then, the velocity $\mathbf{v}(t) = \mathbf{x}'(t)$ exists, and we define the speed of $\mathbf{x}$ to be the magnitude of velocity

$$\text{Speed} = \|\mathbf{v}(t)\|$$

- If $\mathbf{v}$ is itself differentiable, then we call $\mathbf{v}'(t) = \mathbf{x}''(t)$ the acceleration of $\mathbf{x}$ and denote it by $\mathbf{a}(t)$.

Example 4

- Consider the helix

$$\mathbf{x}(t) = (a\cos t, a\sin t, bt), \quad a, b \text{ constants } (a > 0)$$

- Then

$$\begin{aligned} \mathbf{v}(t) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k} \\ \mathbf{a}(t) &= -a\cos t\mathbf{i} - a\sin t\mathbf{j} \end{aligned}$$

The acceleration vector is
parallel to the *xy*-plane (i.e., is horizontal)

- The speed of this helical path is

$$\|\mathbf{v}(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

The speed is constant

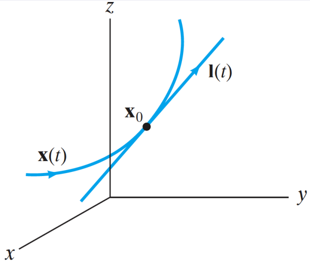
Proposition 1.3: Velocity Vector and Tangent Line

- Let $\mathbf{x}$ be a differentiable path and assume that $\mathbf{v}_0 = \mathbf{v}(t_0) \neq \mathbf{0}$
- Then, a vector parametric equation for the line tangent to $\mathbf{x}$ at $\mathbf{x}_0 = \mathbf{x}(t_0)$ is either

$$\mathbf{l}(s) = \mathbf{x}_0 + s\mathbf{v}_0$$

or

$$\mathbf{l}(t) = \mathbf{x}_0 + (t - t_0)\mathbf{v}_0$$



Example 5

- Let $\mathbf{x}(t) = (3t + 2, t^2 - 7, t - t^2)$
- We find parametric equations for the line tangent to $\mathbf{x}$ at $\mathbf{x}(1) = (5, -6, 0)$

- For this path

$$\mathbf{v}(t) = \mathbf{x}'(t) = 3\mathbf{i} + 2t\mathbf{j} + (1 - 2t)\mathbf{k}$$

- So that

$$\mathbf{v}_0 = \mathbf{v}(t_0) = \mathbf{v}(1) = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

- Thus, b y Proposition 1.3

$$\mathbf{l}(t) = \mathbf{x}_0 + (t - t_0)\mathbf{v}_0 = (5\mathbf{i} - 6\mathbf{j}) + (t - 1)(3\mathbf{i} + 2\mathbf{j} - k)$$

Example 5

- Let $\mathbf{x}(t) = (3t + 2, t^2 - 7, t - t^2)$
- We find parametric equations for the line tangent to $\mathbf{x}$ at $\mathbf{x}(1) = (5, -6, 0)$

$$\mathbf{l}(t) = (5\mathbf{i} - 6\mathbf{j}) + (t - 1)(3\mathbf{i} + 2\mathbf{j} - k)$$

- Taking components, the parametric equations of the tangent line are

$$\mathbf{l}(t) \equiv \begin{cases} x = 3t + 2 \\ y = 2t - 8 \\ z = 1 - t \end{cases} , \quad t \in \mathbb{R}$$